Minimizers of an objective function that generalizes CG and MINRES lie on low dimensional affine subspaces

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Motivation

Hallman and Gu [1] showed that the LSMB iterates
$(A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, x \in \mathbb{R}^n,\ m \geq n,\ K_k = Krylov(A^TA, A^Tb, k),$
$\omega \geq 0)$

$$x_\omega = \underset{x\in K_k}{\arg\min} \|(A^TA + \omega I)^{-1/2}A^T(Ax - b)\|_2$$

are s.t. $(0 \leq \lambda \leq 1)$

$$x_\omega = \lambda x_0 + (1-\lambda)x_\infty \Rightarrow \{x_\omega - x_0\} \subseteq \text{1-dimensional subspace}$$

Proof is complex and does not explain why.

Problem statement

We generalize

$$x_{b,\omega} = \arg\min_{x \in \mathcal{S}} \|(A + \omega I)^{-1/2}(Ax - b)\|$$

for an arbitrary subspace $\mathcal{S}$.

If $\mathcal{S} = Krylov(A, b, k)$

- x_0 is CG solution;
- x_∞ is MINRES solution.

We ask

$$\{x_\omega - x_0 | \omega \geq 0\} \subseteq \text{Low dimensional subspace ?}$$

Index of invariance

We define

$$\textbf{Ind}_A(\mathcal{S}) = \dim(\mathcal{S} + A\mathcal{S}) - \dim(\mathcal{S})$$

- If $\mathcal{S}$ is invariant, $\textbf{Ind}_A(\mathcal{S}) = 0$
- If $\mathcal{S}$ is Krylov (but not invariant), then $\textbf{Ind}_A(\mathcal{S}) = 1$

Let V span $\mathcal{S}$, $\begin{bmatrix} V & V' \end{bmatrix}$ span $\mathcal{S} + A\mathcal{S}$ and V'' the complement, we have the decomposition

$$\begin{bmatrix} V^* \\ V'^* \\ V''^* \end{bmatrix} A \begin{bmatrix} V & V' & V'' \end{bmatrix} = \left[\begin{array}{c|c|c} T & B^* & 0 \\ \hline B & C & D^* \\ \hline 0 & D & E \end{array}\right]$$

Main result

Let A be hermitian and invertible, $\omega > \omega_{min} = -\lambda_{\min}(A)$, $\mathcal{S}$ a subspace,

$$x_{b,\omega} = \arg\min_{x \in \mathcal{S}} \|(A + \omega I)^{-1/2}(Ax - b)\|$$

Then [2]

$$\{x_{b,\omega} - x_{b,0} \mid \omega > \omega_{min}\} \subseteq \text{span}(V(T^*T + B^*B)^{-1}B^*) = U$$

where

$$\dim(U) \leq \mathbf{Ind}_A(\mathcal{S})$$

- U does not change with b;
- If $\mathcal{S}$ invariant: $U = \{0\}$;
- If $\mathcal{S}$ Krylov: $U =$ 1-dimensional.

Weak converse holds

For any $d \in \text{span}(V(T^*T + B^*B)^{-1}B^*)$, there exist b and ω such that

$$d = x_{b,\omega} - x_{b,0}$$

Strong converse does not hold

There exist a matrix A such that for all b

$$\{x_{b,\omega} - x_{b,0} \mid \omega > -\lambda_{\min}(A)\} \subseteq U'$$

where

$$\dim(U') < \mathbf{Ind}_A(\mathcal{S})$$

References

Eric Hallman and Ming Gu.
LSMB: Minimizing the backward error for least-squares problems.
SIAM Journal on Matrix Analysis and Applications, 39(3):1295–1317, 2018.

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In preparation.